COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VI, 73–91 (1953)

Total Reflection of Waves from a Point Source*

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I. *Introduction*

In this paper an analysis is made of the reflection of a pulse of sound from a point source in a non-absorbing liquid plane boundary, under the circumstance that the index of refraction is <1. The analysis is equally applicable to the total reflection of electromagnetic radiation from a vertical dipole, in the absence of absorption. The results clarify the mechanism of propagation of the head wave which has been observed in acoustic experiments. It is shown without approximation that a pulse is received in the first medium at a time corresponding to propagation along the surface with wave velocity of the second medium.

Under certain circumstances a pole in the complex plane of integration makes a contribution to the field at long distances. However the pole always lies on the imaginary axis for non-absorbing media, giving rise to an exponentially decreasing term at long distances. Thus no true (undamped) surface waves can occur. The contribution from this pole is absent in the electromagnetic case when, as is usual, the magnetic permeabilities of the two media are equal.

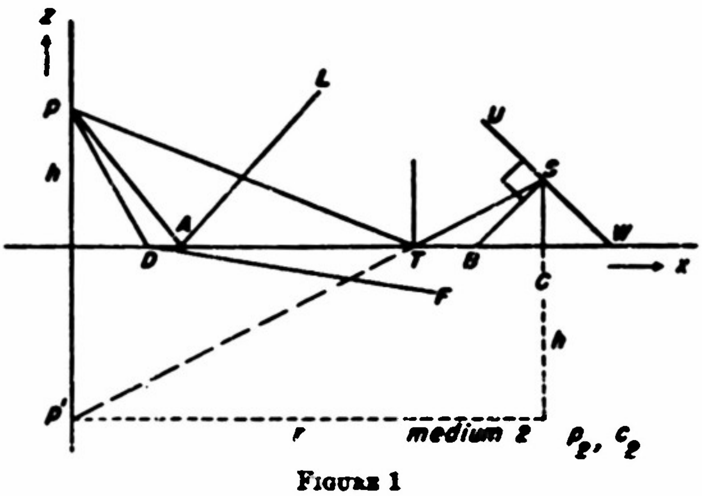

FIGURE 1

Figure 1 shows a point source at P, a height h above an infinite plane

*Contribution from the Scripps Institute of Oceanography. This work represents one of the results of research carried out under contract with the Bureau of Ships, Navy Department.

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surface separating two non-absorbing media. The wave velocity is greater in medium 2 than in medium 1. The signal is received at S, a height z above the bounding surface.

Let the source be harmonic, of circular frequency ω. In the acoustic case, when both media are liquids, the acoustic potentials ϕ_1 and ϕ_2 satisfy in their respective media

$$\Delta\phi_1 + k_1^2\phi_1 = 0, \tag{1}$$
$$\Delta\phi_2 + k_2^2\phi_2 = 0$$

with the boundary conditions, on the plane $z = 0$

$$\rho_1\phi_1 = \rho_2\phi_2\,, \tag{2}$$
$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z}.$$

In these equations ρ denotes density, c wave velocity, and $k = \omega/c$.

In the electromagnetic case, when both media are non-conducting, and the source is a vertical dipole, the Hertz potentials π_1 and π_2 satisfy [1]

$$\Delta\pi_1 + k_1^2\pi_1 = 0, \tag{3}$$
$$\Delta\pi_2 + k_2^2\pi_2 = 0$$

with the boundary conditions, on the plane $z = 0$

$$\frac{k_1^2}{\mu_1}\pi_1 = \frac{k_2^2}{\mu_2}\pi_2\,, \tag{4}$$
$$\frac{\partial\pi_1}{\partial z} = \frac{\partial\pi_2}{\partial z}.$$

In equation (4) μ is the magnetic permeability. It will be noticed that when $\mu_1 = \mu_2$ the electromagnetic problem is less general than the acoustic, since the ratio of the coefficients of π_1 and π_2 in the first of equations (4) is $k_1^2/k_2^2 = c_2^2/c_1^2$, whereas ρ_1/ρ_2 is independent of c_1/c_2.

The field in medium 2 has been discussed previously [2], in this paper the field in medium 1 will be analyzed. In the main our results confirm those of previous investigators [3] but some new features are introduced, to wit: (*1*) The time of arrival of a pulse is evaluated exactly, and shown to agree with the conclusions inferred from the approximate evaluation of the field from a harmonic source. (*2*) The validity of the method of steepest descents in this problem is examined by evaluating the next higher approximation, and a number of the conditions for the validity of the method are shown to have simple and natural physical interpretations. The amplitude dependence of the head wave is also given a simple geometrical interpretation. The head wave denotes the wave which in the region of total reflection arrives before the reflected wave, and may arrive before the directly transmitted wave.[1] (3) Under certain circumstances a pole lying on the imaginary axis in the complex plane of integration makes a contribution to the field, yielding an exponentially damped wave at long distances in medium 1. The circumstances are such that the pole makes no contribution in the electromagnetic case when $\mu_1 = \mu_2$. The contribution of this pole does not seem to have been previously remarked.[2]

II. *Notation and Basic Formulas*

We employ our previous notation: the acoustic potential at S (Figure 1) with the coordinates (x,y,z) is

$$\phi_1 = \phi_0 + \phi_r = \frac{\exp\{ik_1R\}}{R} + \phi_r\,, \tag{5}$$

R is the distance PS (Figure 1), and the reflected wave ϕ_r is

$$\phi_r = ik_1 \int_0^\infty du \frac{u}{\sqrt{1-u^2}} f_r J_0(k_1ru) \exp\{ik_1(z+h)\sqrt{1-u^2}\}, \tag{6}$$

$$f_r = \frac{\beta\sqrt{1-u^2} - \sqrt{\alpha^2-u^2}}{\beta\sqrt{1-u^2} + \sqrt{\alpha^2-u^2}}. \tag{7}$$

In equations (6) and (7), $r = (x^2 + y^2)^{1/2}$, $\alpha = c_1/c_2$ is assumed <1, β equals ρ_2/ρ_1 (acoustic) or $\mu_1\alpha^2/\mu_2$ (electromagnetic), and may have any positive value. Equation (7) for the reflection coefficient f_r is equation (10) of reference 2. Equation (6) follows readily from equation (6) of reference 2 by the procedure used to obtain equation (13) of reference 2. Details of the derivation of equation (6) are also given by Ott [3]. The correct behavior of the solution at infinity in either medium is guaranteed by choosing $\sqrt{\alpha^2 - u^2}$ and $\sqrt{1 - u^2}$

[1]Such early arriving waves have been observed in geophysical prospecting by explosive sounds (cf. Muskat, ref. 3), and, in underwater propagation by J. L. Worzel and M. Ewing, *Explosion sounds in shallow water*, Geol. Soc. of Am., Memoir 27, 1948.

[2]Of the authors listed in reference 3, Lamb solved not the problem under consideration here, but some related problems. Jeffreys uses operational methods throughout. Muskat obtained a valid formal solution, but evaluated the resultant integrals by means of a relatively crude approximation. Ott's analysis parallels our own but he does not follow the steepest descent contours in the complex plane with the care which we have used. In essence our evaluation of the integrals confirms Ott's results, but extends them by including the small but theoretically interesting contributions from the pole. Kruger evaluated the integrals without using the method of steepest descents, for the electromagnetic case. However, his analysis is immediately applicable to total reflection only when $h = 0$, and uses the restrictive assumption that $\mu_1 = \mu_2$. Brehovskih's work, in Russian, seems to parallel ours, but he makes no mention of the pole.

positive real for small positive u and positive imaginary for large positive u. The choice of sign is achieved by drawing the cuts through α and 1 upward, as shown in Figure 2. It will prove convenient to draw the cuts through $-\alpha$ and -1 downward as shown; for the purposes of this section their directions are not significant. The contour of equation (6) runs from 0 to ∞ on the real axis without crossing any of the cuts. For $z+h > 0$

$$\frac{\exp\{ik_1\sqrt{r^2 + (z+h)^2}\}}{\sqrt{r^2 + (z+h)^2}} \tag{8}$$
$$= \int_0^\infty dt \, \frac{t}{\sqrt{t^2 - k_1^2}} J_0(rt) \exp\{-(z+h)\sqrt{t^2 - k_1^2}\}.$$

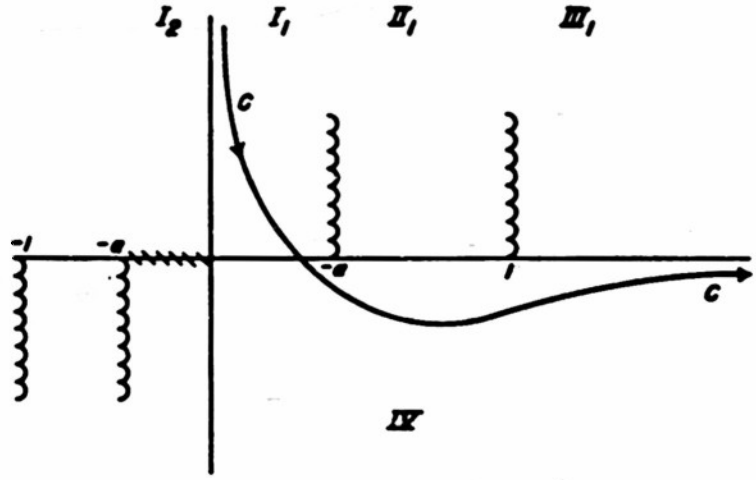

FIGURE 2

Equation (8) is well known; it implies that the real part of $\sqrt{t^2 - k_1^2}$ is always >0 on the contour, (cf. Stratton [5]) or equivalently, that the contour passes below the singularity at $t = k_1$ (cf. Watson [6]). Letting $t = k_1 u$ in equation (8), remembering $k_1 > 0$, and noting that $\sqrt{u^2 - 1} = -i\sqrt{1 - u^2}$ is consistent with the definitions of the signs of the radicals in equations (6) and (8), it can be seen that

$$\phi_1 = \phi_0 + \phi_s + \phi_t, \tag{9}$$

$$\phi_s = \frac{e^{ik_1 d}}{d}\left[\frac{\beta(z+h)/d - \sqrt{\alpha^2 - r^2}}{\beta(z+h)/d + \sqrt{\alpha^2 - r^2}}\right], \tag{10}$$

$$\phi_t = 2ik_1 \int_0^\infty du \, \frac{u}{\sqrt{1-u^2}} F(u) J_0(k_1 r u) \exp\{ik_1(z+h)\sqrt{1-u^2}\}, \tag{11}$$

$$F(u) = \frac{\sqrt{\alpha^2 - \tau^2}}{\beta\sqrt{1 - \tau^2} + \sqrt{\alpha^2 - \tau^2}} - \frac{\sqrt{\alpha^2 - u^2}}{\beta\sqrt{1 - u^2} + \sqrt{\alpha^2 - u^2}}, \tag{12}$$

$$d = \sqrt{r^2 + (z + h)^2}, \qquad \tau = r/d, \qquad \sqrt{1 - \tau^2} = \frac{z + h}{d}. \tag{13}$$

The distance between the receiver and the image of the source is $d = P'S$ (Figure 1); τ is the sine of the angle of incidence TPP' (Figure 1). In equation (7) u equals the sine of the angle of incidence (see [2]); equations (6) and (11) are integrals over real and complex angles of incidence. The amplitude and phase of ϕ_s are precisely those expected for a spherical wave propagating along the geometrically reflected ray PTS (Figure 1), if we assume this wave to be reflected with the plane wave coefficient of equation (7), with $u = \tau$. The sign of $\sqrt{\alpha^2 - \tau^2}$ is identical with that of $\sqrt{\alpha^2 - u^2}$ when $u = \tau$. We infer therefore that all deviations from ray or geometrical propagation are contained[3] in the term ϕ_i.

The poles of the integrand of equation (11) lie at the roots of

$$\beta\sqrt{1 - u^2} + \sqrt{\alpha^2 - u^2} = 0 \tag{14}$$

or at

$$\begin{aligned} u &= \pm\left(\frac{\beta^2 - \alpha^2}{\beta^2 - 1}\right)^{1/2}, && 1 < \beta, \\ u &= \pm i\left(\frac{\beta^2 - \alpha^2}{1 - \beta^2}\right)^{1/2}, && \alpha < \beta < 1, \\ u &= \pm\left(\frac{\alpha^2 - \beta^2}{1 - \beta^2}\right)^{1/2}, && \beta < \alpha. \end{aligned} \tag{15}$$

In that part of the complex plane which is above the real axis, we denote by I the region to the left of the cut at α, by II the region between the cuts at α and 1, and by III the region to the right of the cut at 1. Since the cuts may (and will later) be drawn so that they intersect the positive imaginary axis, we further distinguish between region I_1, containing those points in region I which lie in the first quadrant of the complex plane, and I_2, containing those points in region I which lie in the second quadrant, and similarly for regions II and III. The fourth quadrant is designated by IV. Some of these regions are shown in Figure 2. In regions I_1, I_2, II_1, III_1, and IV the real and imaginary parts of $(1 - u^2)^{1/2}$, $(\alpha^2 - u^2)^{1/2}$, and $u/(1 - u^2)^{1/2}$ are negative in the following

[3]This formulation, in which $\phi_i = 0$ in the geometrical limit $k_1 \to \infty$ appears more logical than the customary procedure of ignoring the dependence on α, β, and angle of incidence of the reflection coefficient, whereby the field in medium 1 is regarded as the sum of spherical waves radiating from sources of equal strength at P and P' (Figure 1), plus a contour integral, which integral is not zero in the geometrical limit. Cf. Stratton, *op. cit.*, p. 573 ff., or Ott, *op. cit.*

regions:

$$
(16)\qquad
\begin{aligned}
\mathcal{R}e\,(1-u^2)^{1/2} &< 0: && \mathrm{III}_1\\
\mathcal{I}m\,(1-u^2)^{1/2} &< 0: && \mathrm{I}_1\,,\ \mathrm{II}_1\\
\mathcal{R}e\,(\alpha^2-u^2)^{1/2} &< 0: && \mathrm{II}_1\,,\ \mathrm{II}_2\,,\ \mathrm{III}_1\\
\mathcal{I}m\,(\alpha^2-u^2)^{1/2} &< 0: && \mathrm{I}_1\,,\ \mathrm{II}_2\\
\mathcal{R}e\,u/(1-u^2)^{1/2} &< 0: && \mathrm{I}_2\,,\ \mathrm{II}_2\,,\ \mathrm{III}_1\\
\mathcal{I}m\,u/(1-u^2)^{1/2} &< 0: && \mathrm{III}_1\,,\ \mathrm{IV}.
\end{aligned}
$$

Otherwise they are positive. The real and imaginary parts of the radicals change sign discontinuously at the cuts but become zero on the real or on the imaginary axis if they change sign crossing it.

Equations (16) show there are no roots of equation (14) in region IV. Consequently, as shown for similar integrals by Muskat and Ott [3] we obtain

$$
(17)\qquad \phi_t = ik_1 \int_C du\, \frac{u}{\sqrt{1-u^2}}\, F(u) H_0^{(1)}(k_1 r u) \exp\{ik_1(z+h)\sqrt{1-u^2}\}.
$$

The contour C of equation (17) runs from $i\infty$ to ∞ on the real axis, as shown in Figure 2. Replacing $H_0^{(1)}$ by its asymptotic expansion, equation (17) becomes

$$
(18)\qquad \phi_t = \left(\frac{2k_1}{\pi r}\right)^{1/2} e^{i\pi/4} \int_C du \left(\frac{u}{1-u^2}\right)^{1/2} F(u) \exp\{ik_1 A(u)\},
$$

$$
(19)\qquad A(u) = (z+h)\sqrt{1-u^2} + ru.
$$

In equation (17) $-\pi < \arg u \leq \pi$. To prevent circling the singularity at the origin, the cut at $-\alpha$ is extended to the origin along the negative real axis as indicated by the stippling in Figure 2.

We evaluate ϕ_t, equation (18), by the method of steepest descents.[4] There are two distinct cases: $\tau < \alpha$, considered in the immediately following section and $\tau > \alpha$, which is deferred to section IV.

III. *Solution for* $\tau < \alpha$

The saddle points of the integral (18) lie at the roots $u = \pm\tau$ of the equation $A'(u) = 0$, with

$$
(20)\qquad A'(u) = r - \frac{(z+h)u}{\sqrt{1-u^2}}.
$$

We shall be concerned only with the point $u = \tau$, which for $\tau < \alpha$ lies in I_1. $A(u)$ is pure real on the real axis between 0 and 1, and, in I_1, has a maximum at $u = \tau$. Along the contour $\mathcal{R}e\, A(u) = A(\tau)$ through $u = \tau$ in I_1, $\mathcal{I}m\, A(u)$

[4] Watson, *op. cit.*, p. 2.

can have no maxima and but a single minimum, at $u = \tau$. Evaluation of ϕ_t, equation (18), by the method of steepest descents now proceeds in the usual way. To completely justify the analysis however it is necessary[5] to make sure that it is possible to deform the contour C of Figure 2 into the contour $\mathcal{R}e\, A(u) = A(\tau)$ passing through $u = \tau$ in I_1.

The argument required is similar to that employed previously. Use is made of equations (16) and (20), together with the remark that the contour is parallel to the imaginary (real) axis if and only if $A'(u)$ is pure real (imaginary). If $u = \rho e^{i\theta}$, then in III_1 and IV, for large ρ

(21) $$A(u) = \rho\{[r \cos \theta - (z + h) \sin \theta] + i[(z + h) \cos \theta + r \sin \theta]\}$$

while in I_1 and I_2, for large ρ

(22) $$A(u) = \rho\{[r \cos \theta + (z + h) \sin \theta] + i[r \sin \theta - (z + h) \cos \theta]\}.$$

It can be seen from equation (16) that the only roots of equation (14) which can possibly concern us lie in region II. We can conclude that the contour C can be deformed into the contour $\mathcal{R}e\, A(u) = A(\tau)$, which we denote by C' (Figure 3), without crossing any poles.

We recall that u is to be interpreted as the sine of the angle of incidence of an arbitrary ray from the source. The equation $u = \tau$ for the saddle point of

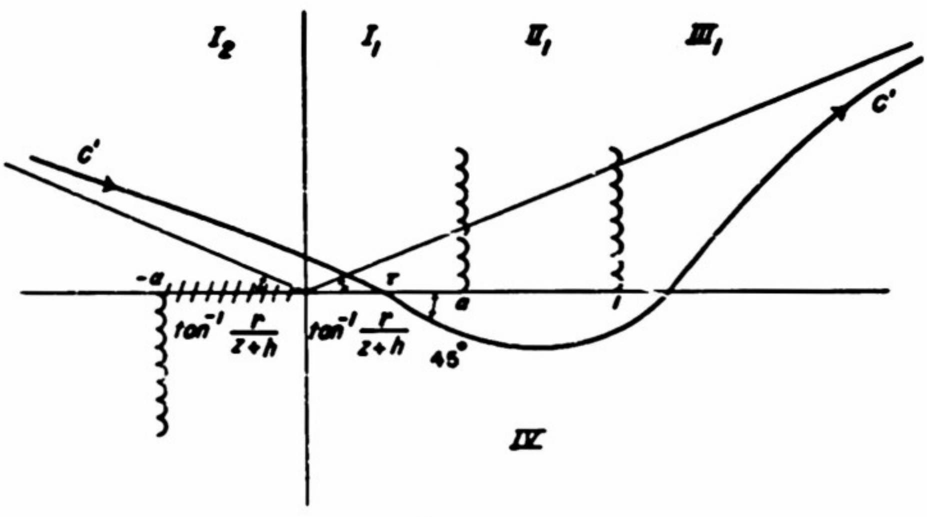

FIGURE 3

equation (18) corresponds therefore to the assertion that the *deviations* from ray propagation are in this case, $\tau < \alpha$, determined primarily by radiation along the direction of the expected geometrical path. This interpretation is confirmed by the fact that $\exp\{ik_1A(\tau)\} = \exp\{ik_1 d\}$ is a phase factor common to ϕ_e, equation (10), and to this estimate of ϕ_t. Moreover since $F(u) = 0$ at $u = \tau$,

[5] This point has not always been sufficiently emphasized.

ϕ_t is seen to be small compared to ϕ_s (in the limit of high frequencies and large distances, where the method of steepest descents is applicable). These results are in complete agreement with those of Ott [3].

IV. *Solution for* $\tau > \alpha$

When $\tau > \alpha$ it is necessary to deform the contour into the shape C_1 followed by C_2 of Figure 4 in order to obtain a convergent contour integral through the saddle point $u = \tau$. We write

$$\phi_t = \phi_t(C_1) + \phi_t(C_2) \tag{23}$$

where ϕ_t is given by equation (17), and both $\phi_t(C_1)$ and $\phi_t(C_2)$ are given by equation (17) with the substitution of C_1 or C_2 for C. Replacing $H_0^{(1)}$ by its asymptotic expansion, $\phi_t(C_2)$ is given by equation (18) with C_2 instead of C.

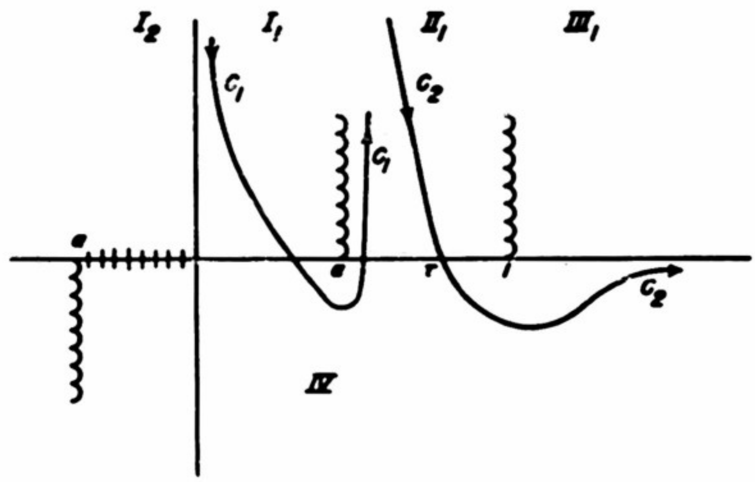

FIGURE 4

Since $\sqrt{\alpha^2 - u^2}$ does not occur in $A(u)$ the argument in the preceding section concerning the determination of the contour $\mathcal{R}e\ A(u) = A(\tau)$ is wholly unchanged, except for the fact that the contour extends to infinity on the left in II_2 instead of I_2. The only question to be settled is whether deformation of the contour requires crossing any of the poles in II. This question will be considered below. Except for possible contributions from poles in II the integral $\phi_t(C_2)$ is seen to yield precisely the same results as did ϕ_t in the preceding section. The contour into which C_2 is deformed is termed C_2'.

We proceed to $\phi_t(C_1)$: The first term in $F(u)$, equation (12), is merely a constant. Referring to equation (17) and to Figure 4, it is evident that in $\phi_t(C_1)$ the contour C_1 can be closed at infinity for this first term in $F(u)$, since the integral containing this term contains no terms involving $\sqrt{\alpha^2 - u^2}$. Nor does it contain any poles. Consequently the first term in $F(u)$ results in a vanishing integral and we have

(24)
$$\phi_i(C_1) = -ik_1 \int_{C_1} du \frac{u}{\sqrt{1-u^2}} \frac{\sqrt{\alpha^2 - u^2}}{\beta\sqrt{1-u^2} + \sqrt{\alpha^2-u^2}} \cdot \exp\{ik_1(z+h)\sqrt{1-u^2}\}H_0^{(1)}(k_1ru).$$

In II_1 the contour C_1 lies along the cut. The contour C_1 can be deformed so as to lie along the cut in I_1, since there are no poles in I_1. Because the integrand in equation (24) vanishes at $u = \alpha$ the integral around a small circle about $u = \alpha$ is zero. It follows that

(25)
$$\phi_i(C_1) = -2ik_1 \int_{C_b} du \frac{u\sqrt{\alpha^2-u^2}}{\beta^2(1-u^2) - (\alpha^2 - u^2)} \cdot \exp\{ik_1(z+h)\sqrt{1-u^2}\}H_0^{(1)}(k_1ru).$$

In equation (25) the contour C_b runs in II_1 along the cut, starting at $u = \alpha$ and going to infinity.

We desire the "best" contour for the branch line integral $\phi_i(C_1)$; namely a contour which appears likely to minimize the error. Although no saddle point can be found for $\phi_i(C_1)$, it is very reasonable to deform C_1 into a contour along which the phase of the integrand remains constant. Such a steepest descent contour, whether or not it passes through a saddle point, still seems least likely to introduce complicating cancellations and reinforcements, particularly in the large k_1r limit (in which we are ultimately going to be most interested).

Consequently an approximate expression for $\phi_i(C_1)$ is obtained much as before by replacing $H_0^{(1)}$ in equation (25) by its asymptotic expansion, and determining the contour $\Re e\, A(u) = A(\alpha)$ into which the cut through α must be deformed. Only if this deformation of the contour requires crossing any of the poles in II must their contributions be considered.[4] The contour $\Re e\, A(u) = A(\alpha)$, which we term C_1', is shown in Figure 5 where the cut through α has been deformed so that it coincides with C_1'. The deformation is justified as in the preceding section. The integral along C_1' is computed, as in the method of steepest descents, by expanding the integrand about the point $u = \alpha$. The final result is an expression for the head wave:

(26)
$$\phi_i(C_1') = \frac{2\alpha i}{k_1\beta(1-\alpha^2)^{1/4}} \frac{e^{ik_1\psi}}{r^{1/2}[r(1-\alpha^2)^{1/2} - (z+h)\alpha]^{3/2}}$$

and

(27)
$$\psi = (z+h)\sqrt{1-\alpha^2} + r\alpha.$$

[4] It is definitely not correct to try to evaluate the integral along an arbitrary contour between α and ∞. The exact value of the integral is independent of the choice of contour, of course. But because the integral is not evaluated exactly, the estimated value of the integral can depend on the choice of contour, as apparently happens in the Sommerfeld problem. T. Kahan and G. Eckart, Jour. Phys. Rad., *10*, 165 (1949) and Phys. Rev., *76*, 406 (1949).

Equation (26) shows that in the limit of high frequencies and large distances the amplitude of $\phi_i(C_1')$ is small compared to ϕ_s, equation (10). This was to be expected, since $\phi_i(C_1')$ is part of the deviation from geometrical propagation.

We return to the problem of whether or not poles are crossed. It is apparent from the above discussion that we might just as well have deformed the contours to C_1' and C_2' before replacing $H_0^{(1)}$ by its asymptotic expansion. Figure 5 illustrates some intermediate stages in the deformation of the contour C of Figure 2 into integrals along C_1' and C_2' and shows the cuts through $u = -\alpha$ and $u = 1$. The contour C_1' coincides with the cut through α and is drawn stippled, as is the cut from $u = -\alpha$ to $u = 0$. The portion of C_2' to the left of $u = \tau$ is also shown. Regions I_1 and I_2 lie below and to the left of C_1'. In Figure 5 the contour C has been bent around into region II between C_1' and C_2', but has not yet been extended to infinity. The dotted line is a deformation of C which has not crossed the imaginary axis.

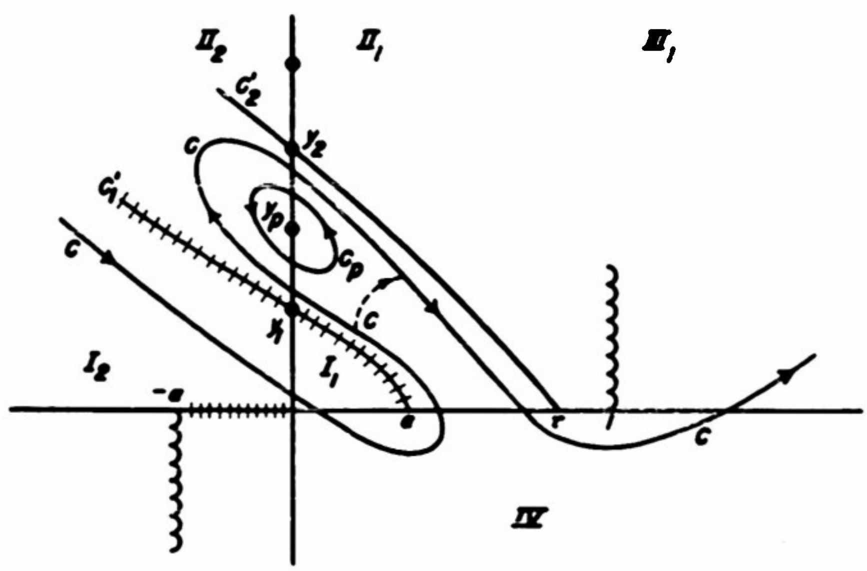

FIGURE 5

Referring to equation (15) it is clear that if $1 < \beta$ it is possible to extend to C in Figure 2 to the dotted C in Figure 5 to the solid C in Figure 5 and so on to infinity in II_1, without crossing any poles. When $\beta < \alpha$ the poles lie on the real axis between $-\alpha$ and α *in region II*. Thus C can again be deformed to infinity in II without crossing a pole. For $\alpha < \beta < 1$ the poles lie on the imaginary axis; the only root of significance is the positive imaginary root in II. Let $u = iy_1$ (iy_2) at the intersection of C_1' (C_2') with the imaginary axis. If the pole lies at $y_p > y_2$ (Figure 5), the contour can obviously be deformed from the dotted C to the solid C without crossing the pole, and so on to infinity. If the ordinate of the pole is $<y_1$, the pole lies below the cut C_1', and is again never crossed. The only case that remains is $\alpha < \beta < 1$, $y_1 < y_p < y_2$, as illustrated in Figure 5. In this case the contour cannot be extended from the dotted C to the solid C without crossing the pole, so that to the integral along the solid C must be added the integral around the closed loop surrounding y_p; this integral readily yields

$$(28)\quad \phi_t(p) = 4ik_1\beta^2 \frac{(1-\alpha^2)^{1/2}}{(1-\beta^2)^{1/2}} \exp\{ik_1[(z+h)(1-\alpha^2)^{1/2}/(1-\beta^2)^{1/2}]\} \cdot K_0[k_1r(\beta^2-\alpha^2)^{1/2}/(1-\beta^2)^{1/2}].$$

$\phi_t(p)$ is the contribution from the pole and in this case where $\alpha < \beta < 1$, $y_1 < y_p < y_2$ we have

$$(29)\quad \phi_t(C_1) = \phi_t(p) + \phi_t(C_1')$$

where $\phi_t(C_1')$ is still given by equation (26). In equation (28), K_0 is the modified Hankel function. The condition $y_1 < y_p < y_2$ can be written as

$$(30)\quad (1-\alpha^2)^{1/2} + \frac{r\alpha}{z+h} < \left(\frac{1-\alpha^2}{1-\beta^2}\right)^{1/2} < \frac{d}{z+h}.$$

We have been unable to interpret $\phi_t(p)$ physically. Since for large x

$$(31)\quad K_0(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}$$

$\phi_t(p)$ decreases exponentially for large r, and therefore can hardly be interpreted as a surface wave in this problem which neglects absorption, despite the $r^{-1/2}$ spreading factor inferred from equations (28) and (31). It is likely that at large distances the magnitude of $\phi_t(p)$ is small compared to the *error* made in deriving equation (26). It is possible that $\phi_t(p)$ is a portion of the difference between the accurate and asymptotic value of $\phi_t(C_1)$, or it may be a spurious result of the analysis. All that can be said is that our choice of "best" contour C_1' leads naturally to $\phi_t(p)$ when $\alpha < \beta < 1$ and equation (30) is satisfied. In the electromagnetic case when $\mu_1 = \mu_2$, we have $\beta = \alpha^2$, so that $\alpha < \beta < 1$ is not possible under these circumstances.

V. *Geometrical Interpretation*

In this section we assume the head wave propagates from P to S along the path $PABS$ (Figure 1) with velocity c_2 along AB, and show that this assumption leads to simple geometrical interpretations of the results of the preceding sections.

In Figure 1

$$(32)\quad AB = r - \frac{(z+h)\alpha}{\sqrt{1-\alpha^2}}.$$

Muskat [3] has pointed out that the phase difference between P and S corresponds to the assumed propagation, i.e., referring to Figure 1, that

$$(33)\quad k_1(PA) + \alpha k_1(AB) + k_1(BS) = k_1\psi$$

holds with ψ given by equation (27). He also remarks that $PABS$ is a path of least time, in the sense of Fermat's principle. The surfaces of constant phase ψ = constant intersect the x,z-plane of Figure 1 in lines parallel to USW and perpendicular to BS. The wave fronts, obtained by rotating USW about the z-axis, are obviously cones with axis OP. These cones have an angle $= \measuredangle CSW =$ the complement of the critical angle.

Consider the two neighboring rays BS and $B'S'$ intersecting the wave front USW (Figure 6). Rotate the section $BSS'B'$ about OP. All the energy

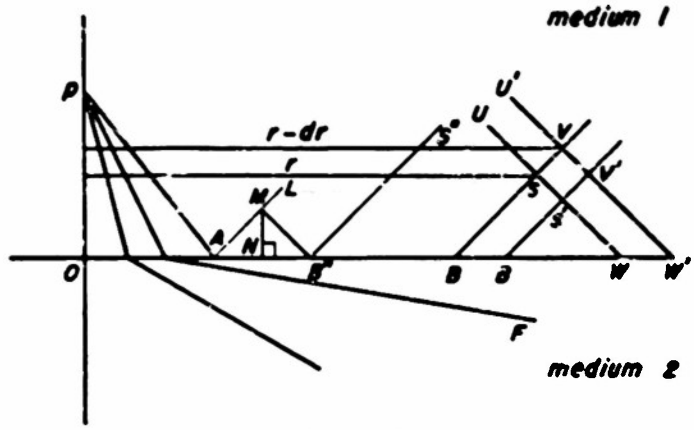

FIGURE 6

lying within the volume formed thereby which in unit time crosses the surface formed by rotating SS' ultimately crosses the surface formed by rotating VV', V and V' being intersections of BS and $B'S'$ with the neighboring plane of constant phase $U'VW'$. Calling the pressure at S $p(r,z)$ and at V $p(r+dr,z+dz)$ and noting that $SS' = VV'$ we have

$$2\pi r p^2(r, z) = 2\pi(r + dr)p^2(r + dr, z + dz). \tag{34}$$

Equation (34) implies that for points on the same ray BS, p is proportional to $r^{-1/2}$ and independent of z. This is precisely the result yielded by equation (26) from which, using equation (32), $\phi_i(C_1')$ is proportional to $(AB)^{-3/2}r^{-1/2}$.

Clearly equation (26) becomes grossly inaccurate[7] as AB approaches zero length, since the solution must be finite at all points other than P. In Figure 6, S'' is a point in the field receiving the head wave via the ray $B''S''$. $B''M$ is the wave front starting at B''. It terminates abruptly at M, since the critical ray PA is reflected into AML and since there is no refracted wave along the surface to contribute to the head wave at points on the surface closer to the origin O than A. We may expect our assumption of propagation along $PABS$ to be valid, and equation (26) to be a good approximation, as long as $B''M$ subtends a large number of Fresnel zones constructed from point S''. In other

[7]Brehovskih, see [3], by a different method obtained instead of equation (26) a modified expression which he claims accurately represents $\phi_i(C_1')$ for small or zero values of AB.

words it is necessary that

$$\frac{S''M - S''B''}{\lambda} \gg 1 \tag{35}$$

when $S''B'' \gg MB''$; equation (35) simplifies to

$$(1 - \alpha^2)(AB)^2 \gg 2\lambda z/\sqrt{1 - \alpha^2}. \tag{36}$$

Equation (26) is the leading term in an asymptotic expansion of $\phi_t(C_1)$, presumably valid in the limit as $k_1r \to \infty$. For finite k_1r the ratio of the next term in the asymptotic expansion to the leading term, equation (26), furnishes an estimate of the error made in using equation (26) to represent the head wave. In[8] equation (25)

$$H_0^{(1)}(k_1ru) = \left(\frac{2}{\pi k_1ru}\right)^{1/2} \exp\{i(k_1ru - \pi/4)\}\left(1 + \frac{\Delta}{8ik_1ru}\right) \tag{37}$$

where $|\Delta| \leq 1$. If in equation (25), we ignore the term in Δ, introduce the new variable η which runs from 0 to ∞ as u traverses C_1' from α to ∞, and expand the terms in the integrand exclusive of the exponential terms in a power series, we obtain

$$\begin{aligned}\phi_t(C_1') = \frac{4i\alpha k_1^{1/2}d^{3/2}}{\pi^{1/2}\beta(1-\alpha^2)^{1/4}} \frac{e^{ik_1 d}}{r^{1/2}[r(1-\alpha^2)^{1/2} - (z+h)\alpha]^{3/2}} \\ \cdot \int_0^\infty d\eta\, e^{-\eta k_1 d}\eta^{1/2}(1 + i\eta\delta + \cdots).\end{aligned} \tag{38}$$

The leading term in equation (38) yields equation (26). In equation (38) $\delta = \sum_j \delta_j$, $j = 1$ to 4, where the δ_j are defined as follows: For $j = 1$ to 3,

$$Q_j = B_j[1 + \delta_j(i\eta) + \cdots] \tag{39}$$

the functions Q_j, $j = 1$ to 3, being respectively $u^{1/2}$, $[\beta^2(1-u^2) - (\alpha^2-u^2)]^{-1}$, and $-idu/d\eta$. δ_4 is defined by

$$(\alpha^2 - u^2)^{1/2} = B_4\eta^{1/2}[1 + \delta_4(i\eta) + \cdots]. \tag{40}$$

In equations (39) and (40) the B_j, $j = 1$ to 4, are numerical coefficients.

In equation (38) the absolute value of the ratio of the term in δ to the leading term is $3\delta/(2k_1d)$, which must be $\ll 1$ if equation (26) is to be accurate. The expression $3\delta/(2k_1d)$ where two or more of the $3\delta_j/(2k_1d)$ are large can sometimes be made $\ll 1$ by cancellation; such cancellation is seldom possible in the omitted term of equation (38), of order η^2 in the integrand, whose ratio to the leading term will be $\sim\delta_j^2(k_1d)^2$. We conclude therefore that it is probably necessary that each of the $3\delta_j/(2k_1d)$ be $\ll 1$, in order that equation (26) be accurate.

The inequality corresponding to $j = 3$ is

$$\frac{3(z + h)}{2(1 - \alpha^2)^{3/2}k_1(AB)^2} \ll 1. \tag{41}$$

[8] Watson, *op. cit.*, p. 219.

It is a confirmation of our geometrical interpretation that equation (41) is equivalent to equation (36). Apparently the geometrical approximations involved in deriving equation (36) implied $z \gg h$. That the right hand side of equation (36) should depend on $(z + h)$ can be inferred from equations (6) and (11) which show that ϕ_r and ϕ_t are functions of z and h only through the sum $(z + h)$.

We may add that in equation (25) the contribution of the term in Δ can be estimated by expending that term in a power series about $u = \alpha$, leading to the inequality $8k_1 r\alpha \gg 1$, a typical condition for the validity of geometrical propagation. Some of the inequalities $3\delta_i/(2k_1 d) \ll 1$ imply

$$\frac{3}{4}\frac{r}{\alpha k_1 (AB)^2} \ll 1. \tag{42}$$

We have not interpreted equation (42). It can be inferred from $3\delta_i/(2k_1 d) \ll 1$ that $(1 - \alpha^2)$ and β must not be too close to zero, conditions to be expected from equation (26), since $\phi_t(C_1)$ cannot increase indefinitely as $(1-\alpha^2)$ and β approach zero.

VI. *Solution for Complex* ω

To this point we have been concerned with positive real k_1 and k_2. To determine the received signal resulting from a pulse, we require the solutions for complex values of ω, where $\mathcal{I}m\ \omega \geq 0$. We retain $k_1 = \omega/c_1$, $k_2 = \omega/c_2$, c_1 and c_2 real. The previously adopted means of obtaining the solution, based on the expansion of e^{ikR}/R in plane waves as first used by Weyl [7], would in the present case[9] involve divergent integrals. We can follow Sommerfeld [8] however; by analytic continuation it can be seen that equation (8) is correct with $(h + z) > 0$, in the extended range $r \geq 0$, $\mathcal{I}m\ k_1 \geq 0$, where $\mathcal{R}e \sqrt{t^2 - k_1^2} \geq 0$. The value of $\sqrt{t^2 - k_1^2}$ at points $t^2 < k_1^2$, k_1 real, is determined by continuation of $\sqrt{t^2 - k_1^2}$ from the upper half plane $\mathcal{I}m\ k_1 > 0$. We note that $\mathcal{R}e \sqrt{t^2 - k_1^2}$ cannot change sign, as k_1 is varied continuously for fixed t, unless $\mathcal{I}m\ k_1$ becomes zero, and that the integral therefore converges, with $\sqrt{t^2 - k_1^2}$ as defined, for $r = 0$ and/or $k_1 = 0$.

Using equation (8), with $(h - z)$ replacing $(h + z)$, so as to represent ϕ_0, equation (5), we can, as does Sommerfeld, obtain a solution for complex k_1. In fact, letting $t = k_1 u$ in equation (8) and the other relevant integrals, the solution in medium 1 is given by equations (5) and (9)–(13), except that the contour for ϕ_t, equation (11), now runs from 0 to ∞ along the line $\arg u = -\arg k_1$. The cuts through $u = 1$ and $u = \alpha$, formerly through $t = k_1$ and αk_1, approach infinity along $\arg u = (\pi/2) - \arg k_1$. The cuts through $u = -1$ and $u = -\alpha$,

[9] The plane wave expansion of e^{ikR}/R appears to us to involve divergent integrals in medium 2 whenever k_2 is complex even with real k_1. When k_2 is complex, for any x and y, there are values of the azimuth angle ψ in equation (12) of reference 2 for which the integrand becomes infinite as θ approaches its upper limit of $\pi/2 - i\infty$. This difficulty does not seem to have been remarked by later writers, e.g. Ott.

which were downward in the t-plane, approach infinity along $\arg u = -(\pi/2) - \arg k_1$. With these provisos, equation (11) can be seen to represent an analytic function of k_1 in the entire range $\mathcal{I}m\, k_1 \geq 0$; as always, $k_2 = \alpha k_1$, $\alpha < 1$. This result is most important. Knowing the solutions are analytic functions of k_1 enables us, as in the next section, to extend integrals over ω into the complex ω-plane. If $k_1 = \rho e^{i\theta}$, $0 < \theta < \pi$, the behavior of ϕ_t, equation (11), for fixed r and z as $\rho \to \infty$ is determined by $\exp \{-\rho(h+z)[\sin \theta\, \mathcal{R}e \sqrt{1 - u^2} + \cos \theta\, \mathcal{I}m \sqrt{1 - u^2}]\}$. The signs of $\mathcal{R}e \sqrt{1 - u^2}$ and $\mathcal{I}m \sqrt{1 - u^2}$ are found by analytic continuation of their values in equation (16), as the cuts of Figure 2 are rotated clockwise. It follows that $\sin \theta\, \mathcal{R}e \sqrt{1 - u^2} + \cos \theta\, \mathcal{I}m \sqrt{1 - u^2}$ is positive for all u on the contour $\arg u = -\arg k_1$, and consequently that equation (11) approaches zero exponentially as $| k_1 | \to \infty$, $0 < \arg k_1 < \pi$. Moreover, for k_1 real, it can be seen that $\phi_t(-k_1) = [\overline{\phi_t(k_1)}]$, the bar denoting complex conjugate. This result was of course to be expected since $\phi_0(-k) = \overline{\phi_0(k)}$ and the differential operator $\Delta^2 + k_1^2$ is real for real k_1.

Using Cauchy's theorem, we may now infer rigorously the result that if $a \geq 0, b \geq 0$,

$$\phi_t(-a + ib) = \overline{\phi_t(a + ib)}. \tag{43}$$

We write

$$\phi_t(k_1) = \frac{1}{2\pi i} \int \frac{dK \phi_t(K)}{K - k_1}, \tag{44}$$

the contour being the real axis from $-\infty$ to ∞ and then around the circle at infinity in the positive $k_1 \equiv K$-plane. Equation (44) is justified by our demonstration that ϕ_t is analytic in the domain, and equation (43) follows at once if we note that the integral over the circle at infinity vanishes since ϕ_t has been shown to become zero exponentially on this circle.

The form of the solution so far obtained is awkward. It is more convenient to introduce $2J_0 = H_0^{(1)} + H_0^{(2)}$ as in Section II. This is not possible if $r = 0$ since $H_0^{(1)}(0)$ is not defined. Restricting ourselves to values[10] of $r > 0$, it is then easy to show using $H_0^{(2)}(k_1 ru) = -H_0^{(1)}(k_1 rue^{i\pi})$ that equation (17) remains valid and is an analytic function of k_1, provided the contour C of equation (17) runs from infinity to the origin along the line $\arg u = \pi - \arg k_1$, and then from the origin to infinity along the line $\arg u = -\arg k_1$. The limits within which the contour can be deformed can be inferred from equations (21) and (22), which remain valid, provided ϑ is interpreted as $\arg k_1 u$ instead of $\arg u$. In III_1 and IV then, the integrand approaches zero exponentially as $u \to \infty$ if

[10]Obviously, in justifying equation (17) (cf. Section II) we should have stated that $r > 0$ but the fact that our formulas involving $H_0^{(1)}$ are not valid for $r = 0$ is of little moment. It is important however that a valid solution from which the fields at $r = 0$ can be evaluated for any k_1 is available (in principle, at any rate), that the analysis for $r > 0$ yields formulas which can be understood and used almost everywhere, and that there is no reason either from the physical properties or from the appearance of the formal solution to expect unusual results on the line $r = 0$.

$(z + h) \cos \theta + r \sin \theta \geq 0$, or it may be deformed from its original $\arg k_1 u = 0$ anywhere within the limits

$$(45) \qquad -\tan^{-1} \frac{z+h}{r} \leq \arg k_1 + \arg u \leq \pi - \tan^{-1} \frac{z+h}{r}.$$

In I_2 we have from equation (22)

$$(46) \qquad \tan^{-1} \frac{z+h}{r} \leq \arg k_1 + \arg u \leq \pi + \tan^{-1} \frac{z+h}{r}.$$

In equations (45) and (46)

$$0 < \tan^{-1} \frac{z+h}{r} < \frac{\pi}{2}.$$

The inequalities hold up to and including the limits; i.e., $\leq$ is correct, if $|k_1| > 0$, as can be seen by substituting the values of $\arg k_1 u$ at the limits in equations (21) and (22). It will be found that $\mathcal{R}e\, A(u)$ is not zero at these values of $\arg k_1 u$, so that equation (17) converges as $\rho \to \infty$.

These results will be used in the following section.

VII. *Reception of a Pulse*

Suppose the wave radiated by the source at P (Figure 1) is given by

$$(47) \qquad \phi_0(t) = \frac{1}{R} f\left(t - \frac{R}{c_1}\right)$$

where R is the distance PS and $f(x) = 0$ for $x < 0$. The function $f(x)$ satisfies sufficient conditions for the existence of a Laplace transform, but is otherwise arbitrary. In particular $f(x)$ satisfies the condition that the integral from 0 to ∞ of $|f(x)|$ exists. Then $f(x)$ can be written in the form

$$(48) \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega x} g(\omega)$$

where $g(\omega)$ is a one-valued regular analytic function of ω in the entire half plane $\mathcal{I}m\, \omega \geq 0$.

It follows, as is well known, that the solution in medium 1 is given by equation (5) where now

$$(49) \qquad \phi_r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} g(\omega) \phi_r(\omega)$$

with $\phi_r(\omega) \equiv \phi_r$ of eq. (6). The functions $\phi_r(\omega)$ have been derived for all $\mathcal{I}m\, \omega \geq 0$ in the preceding section. In particular $\phi_r(\omega)$ may be replaced by equations (9)–(11). The term $\phi_s(\omega)$ yields a wave which is received at a time $t - d/c_1$ and which duplicates in shape as a function of time the shape of the original wave $\phi_0(t)$. This is also a well known result, and is demonstrated by the usual means of closing the contour in the upper ω-plane. We still have to examine equation

$$\phi_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} g(\omega)\phi_s(\omega) \tag{50}$$

which contains all the deviations from geometrical acoustics. From the previous section $\phi_s(\omega)$ is known to be a regular analytic function for any z, r when $\mathscr{I}m\ \omega \geq 0$. Since $g(\omega)$ is also analytic and regular, the contour may be deformed at will in the region $\mathscr{I}m\ \omega \geq 0$.

Deform the contour so that it makes a small half circle about the origin. This is done to avoid the difficulty $k_1 r = 0$. Assume $r > 0$. Then, in equation (50), $\phi_s(\omega)$ is given by equations (17) and (43) with the contour $C(k_1)$ defined as in the previous section, lying within the limits of equations (45) and (46). We shall now show that when $r < a$ the integral, equation (50), is zero over the infinite half circle at infinity in the upper ω-plane, $\mathscr{I}m\ \omega \geq 0$, whenever $t - d/c_1 < 0$. Consequently by Cauchy's theorem $\phi_s(t) = 0$ whenever $t - d/c_1 < 0$, if $r < a$.

We consider first the quarter circle at infinity $0 \leq \arg \omega \leq \pi/2$. For any k_1 in this region, $k_1 = \omega/c_1$, c_1 real, the contours $C(k_1)$ can be deformed to the contour C' of Figure 3. In order that this deformation be possible, according to equations (45) and (46), it is only necessary that

$$-\tan^{-1}\frac{z+h}{r} \leq \arg k_1 + \tan^{-1}\frac{r}{z+h} \leq \pi - \tan^{-1}\frac{z+h}{r} \tag{51}$$

and

$$\tan^{-1}\frac{z+h}{r} \leq \arg k_1 + \pi - \tan^{-1}\frac{r}{z+h} \leq \pi + \tan^{-1}\frac{z+h}{r}. \tag{52}$$

Equations (51) and (52) are satisfied[11] provided $0 \leq \arg k_1 \leq \pi/2$.

If C_ρ represents the contour along the quarter circle of radius ρ, the integral of equation (50), integrated over this contour, can now be written as

$$Q(t) = \frac{1}{2\pi}\int_{C_\rho} d\omega\, e^{-i\omega t} g(\omega)\int_{C'} du\, \frac{u}{\sqrt{1-u^2}} \exp\{ik_1(h+z)\sqrt{1-u^2}\} \cdot F(u) H_0^{(1)}\!\left(\frac{\omega}{c_1} ru\right). \tag{53}$$

The integral over C' in equation (53) converges uniformly as u approaches infinity for any ω on C_ρ. C' is not a function of ω and $g(\omega)$ is bounded. It follows that the order of integration can be interchanged for any finite ρ, and we need only show that

$$\lim_{\rho\to\infty}\frac{1}{2\pi}\int_{C'} du\, \frac{u}{\sqrt{1-u^2}} F(u)\int_{C_\rho} d\omega\, g(\omega) e^{-i\omega t} \cdot \exp\{ik_1(h+z)\sqrt{1-u^2}\} H_0^{(1)}(k_1 ru) = 0. \tag{54}$$

[11] They are not satisfied if $\arg k_1 > \pi/2$; this is the reason why a line of reasoning stemming from equation (43) was adopted.

As $\rho \to \infty$, $H_0^{(1)}(k_1 ru)$ can be replaced by its asymptotic expansion, for any u on C'. $uF(u)/\sqrt{1-u^2}$ is bounded. The limit as $\rho \to \infty$ is therefore determined by the behavior of

$$B(u) = ik_1[A(u) - c_1 t] \tag{55}$$

with $A(u)$ defined by equation (19). Letting $k_1 = \rho e^{i\theta}$, it is evident that the limit approaches 0 exponentially as $\rho \to \infty$ provided

$$-\sin\theta \, \mathcal{R}e \, A(u) - \cos\theta \, \mathcal{I}m \, A(u) + c_1 t \sin\theta < 0 \tag{56}$$

for all $0 \leq \theta \leq \pi/2$. But however, in Section III the contour C' was so defined that $A(u) = d + i\eta$, where $\eta \geq 0$ for all u on C'. Consequently equation (56) is satisfied[13] whenever $-d + c_1 t < 0$. On the other hand whenever $-d + c_1 t > 0$, there will be a continuous set of values of u on C' for which, for a finite range of angles θ, equation (56) is not satisfied. It follows, using equation (43), that the contour can be closed in the upper half ω-plane whenever $t < d/c_1$, and cannot whenever $t > d/c_1$. In other words $\phi_r(t)$ of equation (49), like the geometrically reflected $\phi_s(t)$, begins at $t = d/c_1$ and not before.

When $r > \alpha$ the contour $C(k_1)$ can be deformed into the contours C_1 and C_2 of Figure 4 and finally into the contours C_1' and C_2' of Figure 5, since the asymptotic limits of C_1' and C_2' are the same as C', so that equations (51) and (52) remain relevant. The integrals continue to be uniformly convergent, and the argument goes just as above. The integral along C_2' leads to the same conclusion, namely that the signal is received when $t = d/c_1$. The integral along C_1' leads to equation (56) where in this case $\mathcal{R}e \, A(u) = r\alpha + (z+h)\sqrt{1-\alpha^2}$, so that along this path the signal is received only when

$$t = \frac{r\alpha + (z+h)\sqrt{1-\alpha^2}}{c_1} = \frac{PA + SB}{c_1} + \frac{AB}{c_2} \qquad \text{(Figure 1).}$$

That no poles are crossed deforming the contours when $r < \alpha$, and that the pole is crossed when $r > \alpha$ only when $\alpha < \beta < 1$ and equation (30) is satisfied is shown just as in Sections III and IV. When $\phi_t(p)$ contributes, K_0 is replaced by its asymptotic expansion in equation (28) and it is found that $\phi_t(p)$ does not contribute until $c_1 t$ equals $(z+h)(1-\alpha^2)^{1/2}/(1-\beta^2)^{1/2}$. That this time lies between the times of appearance of the head wave and the directly reflected wave follows from equation (30). Because this time involves β, no simple geometrical interpretation of $\phi_t(p)$ is possible.

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